3-5 Solving Systems of Equations in Three Variables

Main Ideas

- Solve systems of linear equations in three variables.
- Solve real-world problems using systems of linear equations in three variables.

New Vocabulary

ordered triple

GET READY for the Lesson

At the 2004 Summer Olympics in Athens, Greece, the United States won 103 medals. They won 6 more gold medals than bronze and 10 more silver medals than bronze.

You can write and solve a system of three linear equations to determine how many of each type of medal the U.S. Olympians won. Let g represent the number of gold medals, let s represent the number of silver medals, and let b represent the number of bronze medals.

$g + s + b = 103$ U.S. Olympians won a total of 103 medals.

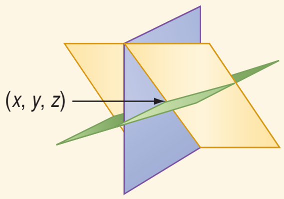
$g = b + 6$ They won 6 more gold medals than bronze.

$s = b + 10$ They won 10 more silver medals than bronze.

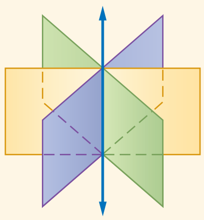
Systems in Three Variables The system of equations above has three variables. The graph of an equation in three variables, all to the first power, is a plane. The solution of a system of three equations in three variables can have one solution, infinitely many solutions, or no solution.

KEY CONCEPT — *System of Equations in Three Variables*

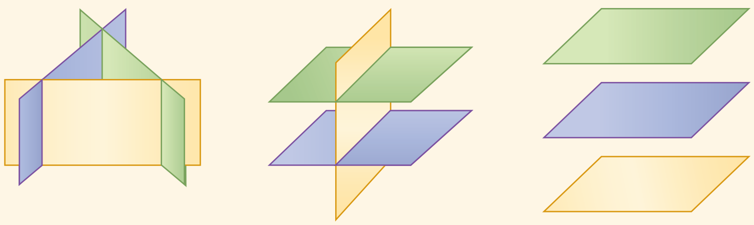
One Solution
- planes intersect in one point



Infinitely Many Solutions
- planes intersect in a line
- planes intersect in the same plane



No Solution
- planes have no point in common



Ruben Sprich/Reuters/CORBIS

Solving systems of equations in three variables is similar to solving systems of equations in two variables. Use the strategies of substitution and elimination. The solution of a system of equations in three variables x, y, and z is called an **ordered triple** and is written as (x, y, z).

EXAMPLE 1 One Solution

Solve the system of equations.

$x + 2y + z = 10$

$2x - y + 3z = -5$

$2x - 3y - 5z = 27$

Step 1 Use elimination to make a system of two equations in two variables.

$x + 2y + z = 10$ → Multiply by 2. → $2x + 4y + 2z = 20$

$2x - y + 3z = -5$ → $(-)\ 2x - y + 3z = -5$

$5y - z = 25$ Subtract to eliminate x.

$2x - y + 3z = -5$ Second equation

$(-)\ 2x - 3y - 5z = 27$ Third equation

$2y + 8z = -32$ Subtract to eliminate x.

Notice that the x terms in each equation have been eliminated. The result is two equations with the same two variables y and z.

Step 2 Solve the system of two equations.

$5y - z = 25$ → Multiply by 8. → $40y - 8z = 200$

$2y + 8z = -32$ → $(+)\ 2y + 8z = -32$

$42y = 168$ Add to eliminate z.

$y = 4$ Divide by 42.

Use one of the equations with two variables to solve for z.

$5y - z = 25$ Equation with two variables

$5(4) - z = 25$ Replace y with 4.

$20 - z = 25$ Multiply.

$z = -5$ Simplify.

The result is $y = 4$ and $z = -5$.

Step 3 Solve for x using one of the original equations with three variables.

$x + 2y + z = 10$ Original equation with three variables

$x + 2(4) + (-5) = 10$ Replace y with 4 and z with -5.

$x + 8 - 5 = 10$ Multiply.

$x = 7$ Simplify.

The solution is $(7, 4, -5)$. Check this solution in the other two original equations.

Study Tip

Elimination Remember that you can eliminate any of the three variables.

CHECK Your Progress

1A. $2x - y + 3z = -2$
$x + 4y - 2z = 16$
$5x + y - 1z = 14$

1B. $3x + y + z = 0$
$-x + 2y - 2z = -3$
$4x - y - 3z = 9$

EXAMPLE Infinitely Many Solutions

2 **Solve the system of equations.**

$$4x - 6y + 4z = 12$$
$$6x - 9y + 6z = 18$$
$$5x - 8y + 10z = 20$$

Eliminate x in the first two equations.

$4x - 6y + 4z = 12$	Multiply by 3.	$12x - 18y + 12z = 36$	
$6x - 9y + 6z = 18$	Multiply by −2.	$(+)\ -12x + 18y - 12z = -36$	Add the equations.
		$0 = 0$	

The equation $0 = 0$ is always true. This indicates that the first two equations represent the same plane. Check to see if this plane intersects the third plane.

$4x - 6y + 4z = 12$	Multiply by 5.	$20x - 30y + 20z = 60$	
$5x - 8y + 10z = 20$	Multiply by −2.	$(+)\ -10x + 16y - 20z = -40$	Add the equations.
		$10x - 14y = 20$	Divide by the GCF, 2.
		$5x - 7y = 10$	

The planes intersect in a line. So, there are an infinite number of solutions.

CHECK Your Progress

2A. $8x + 12y - 24z = -40$
$3x - 8y + 12z = 23$
$2x + 3y - 6z = -10$

2B. $3x - 2y + 4z = 8$
$-6x + 4y - 8z = -16$
$x + 2y - 4z = 4$

Study Tip

Common Misconception

Not every ordered triple is a solution of a system in three variables with an infinite number of solutions. The solution set contains an infinite number of ordered triples but not every ordered triple.

EXAMPLE No Solution

3 **Solve the system of equations.**

$$6a + 12b - 8c = 24$$
$$9a + 18b - 12c = 30$$
$$4a + 8b - 7c = 26$$

Eliminate a in the first two equations.

$6a + 12b - 8c = 24$	Multiply by 3.	$18a + 36b - 24c = 72$	
$9a + 18b - 12c = 30$	Multiply by 2.	$(-)\ 18a + 36b - 24c = 60$	Subtract the equations.
		$0 = 12$	

The equation $0 = 12$ is never true. So, there is no solution of this system.

CHECK Your Progress

3A. $8x + 4y - 3z = 7$
$4x + 2y - 6z = -15$
$10x + 5y - 15z = -25$

3B. $4x - 3y - 2z = 8$
$x + 5y + 3z = 9$
$-8x + 6y + 4z = 2$

Real-World Problems When solving problems involving three variables, use the four-step plan to help organize the information.

Real-World EXAMPLE — Write and Solve a System of Equations

4 **INVESTMENTS** **Andrew Chang has $15,000 that he wants to invest in certificates of deposit (CDs). For tax purposes, he wants his total interest per year to be $800. He wants to put $1000 more in a 2-year CD than in a 1-year CD and invest the rest in a 3-year CD. How much should Mr. Chang invest in each type of CD?**

Number of Years	1	2	3
Rate	3.4%	5.0%	6.0%

Real-World Link

A certificate of deposit (CD) is a way to invest your money with a bank. The bank generally pays higher interest rates on CDs than savings accounts. However, you must invest your money for a specific time period, and there are penalties for early withdrawal.

Explore Read the problem and define the variables.

a = the amount of money invested in a 1-year certificate

b = the amount of money in a 2-year certificate

c = the amount of money in a 3-year certificate

Plan Mr. Chang has $15,000 to invest.

$a + b + c = 15{,}000$

The interest he earns should be $800. The interest equals the rate times the amount invested.

$0.034a + 0.05b + 0.06c = 800$

There is $1000 more in the 2-year certificate than in the 1-year certificate.

$b = a + 1000$

Solve Substitute $b = a + 1000$ in each of the first two equations.

$a + (a + 1000) + c = 15{,}000$ Replace b with $(a + 1000)$.

$2a + 1000 + c = 15{,}000$ Simplify.

$2a + c = 14{,}000$ Subtract 1000 from each side.

$0.034a + 0.05(a + 1000) + 0.06c = 800$ Replace b with $(a + 1000)$.

$0.034a + 0.05a + 50 + 0.06c = 800$ Distributive Property

$0.084a + 0.06c = 750$ Simplify.

Now solve the system of two equations in two variables.

$2a + c = 14{,}000$ Multiply by 0.06. → $0.12a + 0.06c = 840$

$0.084a + 0.06c = 750$ → $(-)\ 0.084a + 0.06c = 750$

$0.036a = 90$

$a = 2500$

Substitute 2500 for a in one of the original equations.

$b = a + 1000$ Third equation

$= 2500 + 1000$ $a = 2500$

$= 3500$ Add.

Substitute 2500 for a and 3500 for b in one of the original equations.

$a + b + c = 15{,}000$ First equation

$2500 + 3500 + c = 15{,}000$ $a = 2500$, $b = 3500$

$6000 + c = 15{,}000$ Add.

$c = 9000$ Subtract 6000 from each side.

So, Mr. Chang should invest $2500 in a 1-year certificate, $3500 in a 2-year certificate, and $9000 in a 3-year certificate.

Check Is the answer reasonable? Have all the criteria been met?

The total investment is $15,000.

$2500 + 3500 + 9000 = 15{,}000$ ✓

The interest earned will be $800.

$0.034(2500) + 0.05(3500) + 0.06(9000) = 800$

$85 + 175 + 540 = 800$ ✓

There is $1000 more in the 2-year certificate than the 1-year certificate.

$3500 = 2500 + 1000$ ✓ The answer is reasonable.

CHECK Your Progress

4. BASKETBALL Macario knows that he has scored a total of 70 points so far this basketball season. His coach told him that he has scored 37 times, but Macario wants to know how many free throws, field goals, and three pointers he has made. The sum of his field goals and three pointers equal twice the number of free throws minus two. How many free throws, field goals, and three pointers has Macario made?

Concepts in Motion
Interactive Lab
algebra2.com

Online Personal Tutor at algebra2.com

CHECK Your Understanding

Examples 1–3 (pp. 146–147)

Solve each system of equations.

1. $x + 2y = 12$
$3y - 4z = 25$
$x + 6y + z = 20$

2. $9a + 7b = -30$
$8b + 5c = 11$
$-3a + 10c = 73$

3. $r - 3s + t = 4$
$3r - 6s + 9t = 5$
$4r - 9s + 10t = 9$

4. $2r + 3s - 4t = 20$
$4r - s + 5t = 13$
$3r + 2s + 4t = 15$

5. $2x - y + z = 1$
$x + 2y - 4z = 3$
$4x + 3y - 7z = -8$

6. $x + y + z = 12$
$6x - 2y - z = 16$
$3x + 4y + 2z = 28$

Example 4 (pp. 148–149)

COOKING For Exercises 7 and 8, use the following information.
Jambalaya is a Cajun dish made from chicken, sausage, and rice. Simone is making a large pot of jambalaya for a party. Chicken costs $6 per pound, sausage costs $3 per pound, and rice costs $1 per pound. She spends $42 on 13.5 pounds of food. She buys twice as much rice as sausage.

7. Write a system of three equations that represents how much food Simone purchased.

8. How much chicken, sausage, and rice will she use in her dish?

Exercises

HOMEWORK HELP	
For Exercises	See Examples
9–19	1–3
20–23	4

Solve each system of equations.

9. $2x - y = 2$
$3z = 21$
$4x + z = 19$

10. $-4a = 8$
$5a + 2c = 0$
$7b + 3c = 22$

11. $5x + 2y = 4$
$3x + 4y + 2z = 6$
$7x + 3y + 4z = 29$

12. $8x - 6z = 38$
$2x - 5y + 3z = 5$
$x + 10y - 4z = 8$

13. $4a + 2b - 6c = 2$
$6a + 3b - 9c = 3$
$8a + 4b - 12c = 6$

14. $2r + s + t = 14$
$-r - 3s + 2t = -2$
$4r - 6s + 3t = -5$

15. $3x + y + z = 4$
$2x + 2y + 3z = 3$
$x + 3y + 2z = 5$

16. $4a - 2b + 8c = 30$
$a + 2b - 7c = -12$
$2a - b + 4c = 15$

17. $9x - 3y + 12z = 39$
$12x - 4y + 16z = 52$
$3x - 8y + 12z = 23$

18. The sum of three numbers is 20. The second number is 4 times the first, and the sum of the first and third is 8. Find the numbers.

19. The sum of three numbers is 12. The first number is twice the sum of the second and third. The third number is 5 less than the first. Find the numbers.

Real-World Link

In 2005, Katie Smith became the first person in the WNBA to score 5000 points.

Source: www.wnba.com

BASKETBALL For Exercises 20 and 21, use the following information.
In the 2004 season, Seattle's Lauren Jackson was ranked first in the WNBA for total points and points per game. She scored 634 points making 362 shots, including 3-point field goals, 2-point field goals, and 1-point free throws. She made 26 more 2-point field goals than free throws.

20. Write a system of equations that represents the number of goals she made.

21. Find the number of each type of goal she made.

FOOD For Exercises 22 and 23, use the following information.
Maka loves the lunch combinations at Rosita's Mexican Restaurant. Today however, she wants a different combination than the ones listed on the menu.

22. Assume that the price of a combo meal is the same price as purchasing each item separately. Find the price for an enchilada, a taco, and a burrito.

23. If Maka wants 2 burritos and 1 enchilada, how much should she plan to spend?

24. **TRAVEL** Jonathan and members of his Spanish Club are going to Costa Rica. He purchases 10 traveler's checks in denominations of \$20, \$50, and \$100, totaling \$370. He has twice as many \$20 checks as \$50 checks. How many of each denomination of traveler's checks does he have?

EXTRA PRACTICE
See pages 896, 928.

Math Online
Self-Check Quiz at algebra2.com

Solve each system of equations.

25. $6x + 2y + 4z = 2$
$3x + 4y - 8z = -3$
$-3x - 6y + 12z = 5$

26. $r + s + t = 5$
$2r - 7s - 3t = 13$
$\frac{1}{2}r - \frac{1}{3}s + \frac{2}{3}t = -1$

27. $2a - b + 3c = -7$
$4a + 5b + c = 29$
$a - \frac{2b}{3} + \frac{c}{4} = -10$

Andy Lyons/Allsport/Getty Images

H.O.T. Problems

28. OPEN ENDED Write an example of a system of three equations in three variables that has (−3, 5, 2) as a solution. Show that the ordered triple satisfies all three equations.

29. REASONING Compare and contrast solving a system of two equations in two variables to solving a system of equations of three equations in three variables.

30. FIND THE ERROR Melissa is solving the system of equations $r + 2s + t = 3$, $2r + 4s + 2t = 6$, and $3r + 6s + 3t = 12$. Is she correct? Explain.

$$\begin{aligned} r + 2s + t = 3 \quad &\rightarrow \quad 2r + 4s + 2t = 6 \\ 2r + 4s + 2t = 6 \quad &\rightarrow \quad (-)\underline{2r + 4s + 2t = 6} \\ & \qquad\qquad\qquad 0 = 0 \end{aligned}$$

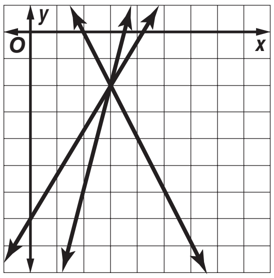
The second equation is a multiple of the first, so they are the same plane. There are infinitely many solutions.

31. CHALLENGE The general form of an equation for a parabola is $y = ax^2 + bx + c$, where (x, y) is a point on the parabola. If three points on the parabola are (0, 3), (−1, 4), and (2, 9), determine the values of a, b, c. Write the equation of the parabola.

32. ***Writing in Math*** Use the information on page 145 to explain how you can determine the number and type of medals 2004 U.S. Olympians won in Athens. Demonstrate how to find the number of each type of medal won by the U.S. Olympians and describe another situation where you can use a system of three equations in three variables to solve a problem.

STANDARDIZED TEST PRACTICE

33. ACT/SAT The graph depicts which system of equations?



A $y + 14 = 4x$
$y = 4 - 2x$
$-7 = y - \frac{5}{3}x$

B $y + 14x = 4$
$-2y = 4 + y$
$-7 = y - \frac{5}{3}x$

C $y - 14 = 4x$
$y = 4 + 2x$
$-7 = y + \frac{5}{3}$

D $y - 14x = 4$
$2x = 4 + y$
$7 = y - \frac{5}{3}x$

34. REVIEW What is the solution to the system of equations shown below?

$$\begin{cases} x - y + z = 0 \\ -5x + 3y - 2z = -1 \\ 2x - y + 4z = 11 \end{cases}$$

F (0, 3, 3)

G (2, 5, 3)

H no solution

J infinitely many solutions

Spiral Review

35. MILK The Yoder Family Dairy produces at most 200 gallons of skim and whole milk each day for delivery to large bakeries and restaurants. Regular customers require at least 15 gallons of skim and 21 gallons of whole milk each day. If the profit on a gallon of skim milk is \$0.82 and the profit on a gallon of whole milk is \$0.75, how many gallons of each type of milk should the dairy produce each day to maximize profits? (Lesson 3-4)

Solve each system of inequalities by graphing. (Lesson 3-3)

36. $y \leq x + 2$
$y \geq 7 - 2x$

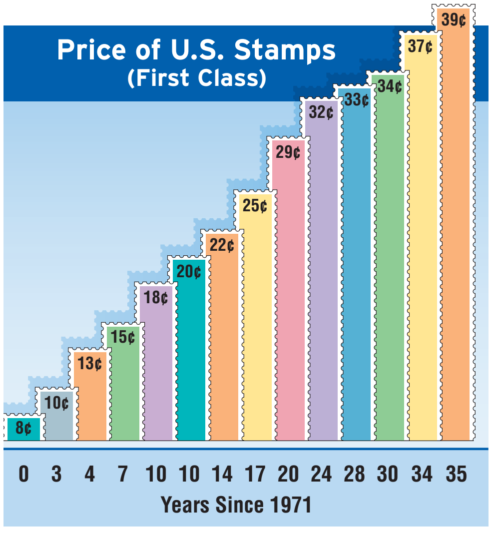
37. $4y - 2x > 4$
$3x + y > 3$

38. $3x + y \geq 1$
$2y - x \leq -4$

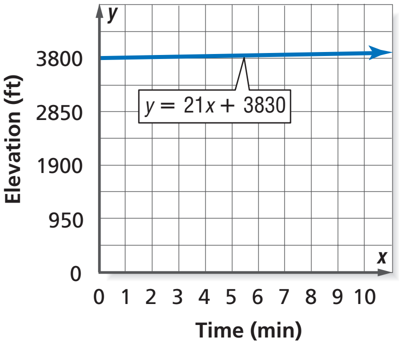
ANALYZE GRAPHS For Exercises 39 and 40, use the following information.
The table shows the price for first-class stamps since July 1, 1971. (Lesson 2-5)

39. Write a prediction equation for this relationship.

40. Predict the price for a first-class stamp issued in the year 2015.



41. HIKING Miguel is hiking on the Alum Cave Bluff Trail in the Great Smoky Mountains. The graph represents Miguel's elevation y at each time x. At what elevation did Miguel begin his climb? How is that represented in the equation? (Lesson 2-4)



Find each value if $f(x) = 6x + 2$ and $g(x) = 3x^2 - x$. (Lesson 2-1)

42. $f(-1)$

43. $f\left(\frac{1}{2}\right)$

44. $g(1)$

45. $g(-3)$

46. TIDES Ocean tides are caused by gravitational forces exerted by the Moon. Tides are also influenced by the size, boundaries, and depths of ocean basins and inlets. The highest tides on Earth occur in the Bay of Fundy in Nova Scotia, Canada. During the middle of the tidal range, the ocean shore is 30 meters from a rock bluff. The tide causes the shoreline to advance 8 meters and retreat 8 meters throughout the day. Write and solve an equation describing the maximum and minimum distances from the rock bluff to the ocean during high and low tide. (Lesson 1-4)